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Out of Equilibrium Thermal Field Theories - Finite Time after Switching on the Interaction - Fourier Transforms of the Projected Functions

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Abstract

We study out of equilibrium thermal field theories with switching on the interaction occurring at finite time by Fourier transforms (also in the relative time s_0) of two-point functions.

To develop a calculation scheme based on first principles, we define a very useful concept of projected functions: a two-point function with the property that it is zero for $x_o < t_i$ and for $y_0 < t_i$; the function depends only on $x_0 - y_0$ for $t_i < x_0$ and $t_i < y_0$. Many important functions are of this type: bare propagators, one-loop self-energies, resummed Schwinger-Dyson series with one-loop self-energies, etc.

For Fourier transforms we define the particular analyticity assumptions: (1) The function of p_0 is analytic above (for a retarded function, below for an advanced function) real axis. (2) The function goes to zero as $|p_0|$ approaches infinity in the upper semiplane.

Without the need to perform the gradient expansion, we obtain the convolution product of projected functions. For bare propagators and self-energies being projected functions satisfying assumptions (1) and (2), we obtain the resummed Schwinger-Dyson series.

The Feynman diagram technique is reformulated: there is no explicit energy conservation at vertices, there is an overall energy-smearing factor taking care of the finite elapsed time (X_0) and the uncertainty relations.

The relation between the amplitudes of the theories with $t_i \to -\infty$ and with t_i finite enables one to rederive the results, such as the cancellation of pinching singularities, the cancellation of collinear and infrared singularities, HTL resummation, etc.

Relaxation phenomena enter through the assumed singularities in the upper semiplane of the first Riemann sheet (for retarded functions, in the lower semiplane for advanced functions). As expected, these singularities contribute terms dying out exponentially with "time" (slow Wigner variable).

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1 Introduction

Out of equilibrium thermal field theory [1, 2] has recently attracted considerable interest [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

In many applications one considers the properties of the Green functions of almost equilibrated systems, infinite time after switching on the interaction. A recent approach based on first principles has been successful in demonstrating the cancellation of collinear [13, 14] and pinching singularities [15, 16, 17, 18, 19, 20], the extension of the hard thermal loop (HTL) approximation [21, 22, 23] to out of equilibrium [24, 25], and applications to heavy-ion collisions [26, 27, 28]. A weak point of the approach was that most of the results were obtained under the assumption that the variation of slow Wigner variables could be ignored or, in other words, that these results were valid only in the lowest order of the gradient expansion [29, 30, 31].

For some problems, e.g., heavy-ion collisions, both limitations are undesired. One would like to consider large deviations from equilibrium. One cannot wait an infinite time as, after a very short time, these systems go apart, probably without reaching the stage of equilibrium. In nuclear collisions, short-time scale features have been studied in a number of papers [32, 33, 34, 35, 36, 37].

Let us start with the remark that $t_i \to -\infty$ is the limit. The intention of such a limit is in fact that also $t - t_i \to \infty$ for all finite times t in the theory. As such, it implies that any information obtained by naive extrapolation to early times[17, 19] is either deformed or lost. It is clear that there is such a piece of information; relaxation phenomena include many processes that are expected to terminate in the limit $t - t_i \to \infty$! Thus one can expect that the full theory also describes $t - t_i$ finite.

In many papers, problems with finite switching-on time[9], especially those related to the inflatory phase of the Universe (see Ref. [38] for further references) are studied by Fourier-transforming in $\vec{s}$ and studying the s_0 dependence directly. In such an approach, the powerful Fourier-transform technique is not sufficiently exploited. Instead, one is relies heavily on differential equations and numerical methods.

In an attempt to remove weak points in both cases, we suggest the application of Fourier techniques (also in s_0) to the case of switching on the interaction at finite time ($t_i = 0$).

The integration path C is now from $t_i + i\epsilon$ to $t_f + i\epsilon$, from $t_f + i\epsilon$ to $t_f - i\epsilon$, and, finally, from $t_f - i\epsilon$ to $t_i - i\epsilon$ (in the rest of the paper the switching-off time is pushed to infinity, $t_f \to +\infty$). As $t_i \to -\infty$, the connection to the Keldysh integration path is established. It comes out that this connection is highly nontrivial.

The fact that $t_i = 0$ limits all times in the perturbation expansion to $t > 0$. For two-point functions after turning to Wigner variables, the slow variable is limited by $X_0 > 0$ and the relative variable is limited by $-2X_0 < s_0 < 2X_0$. This property leads us to the concept of projected function (truncated, "mutilated" function [40], PF in further text). In this paper, the projected function is a very special two-point function $F(x, y) = F(X + s/2, X - s/2)$: it is a function of $(s_0, \vec{s})$ within the interval $-2X_0 < s_0 < 2X_0$ and identical to zero outside. The function F is obtained by applying the projection operator, defined as $= \Theta(2X_0 - s_0)\Theta(2X_0 + s_0)$, which multiplies any function of s_0 defined on the infinite range ($-\infty < s_0 < \infty$), and identical to F within $-2X_0 < s_0 < 2X_0$.

Analogously, the Fourier transform of the projected function (FTPF) is obtained from the Fourier transform of the function defined on the infinite range of s_0 ($-\infty < s_0 < \infty$) with the help of the projection operators P_{X_0}, which are the Fourier transforms of the above given Θ's.

$$G(p_0, \vec{p}; X_0) = \int_{-\infty}^{\infty} dp_0' P_{X_0}(p_0, p_0') G_\infty(p_0', \vec{p}), \tag{1.1}$$

As an illustration, the Fourier transform of the convolution product, Eq. (4.1),

$$C = A * B \Leftrightarrow C(x, y) = \int dz A(x, z) B(z, y), \tag{1.2}$$

of two two-point functions is given by the gradient expansion (note that we have assumed the homogeneity in space coordinates, which excludes any dependence on $\vec{X}$):

$$C_{X_0}(p_0, \vec{p}) = e^{-i\diamondsuit} A_{X_0}(p_0, \vec{p}) B_{X_0}(p_0, \vec{p}), \quad \diamondsuit = \frac{1}{2}(\partial^A_{X_0} \partial^B_{p_0} \cdot - \partial^A_{p_0} \partial^B_{X_0}). \tag{1.3}$$

This general expression should be contrasted with our result valid for A and B being projected functions:

$$\begin{aligned} C_{X_0}(p_0, \vec{p}) &= \int dp_{01} dp_{02} P_{X_0}(p_0, \frac{p_{01} + p_{02}}{2}) \frac{1}{2\pi} \frac{i e^{-iX_0(p_{01} - p_{02})}}{p_{01} - p_{02} + i\epsilon} A_\infty(p_{01}, \vec{p}) B_\infty(p_{02}, \vec{p}), \\ P_{X_0}(p_0, p_0') &= \frac{1}{\pi} \frac{\sin(2X_0(p_0 - p_0'))}{p_0 - p_0'}. \end{aligned} \tag{1.4}$$

For further analysis, the analytic properties of FTFP as a function of complex energy are very important. We define the following properties: (1) the function of p_0 is analytic above (below) the real axis, (2) the function goes to zero as $|p_0|$ approaches infinity in the upper (lower) semiplane. The choice above (below) and upper (lower) refers to retarded (advanced) components (note here that we do not require such properties for the Keldysh components).

Under the assumption that A or B satisfy (1) and (2) (A as advanced ; B as retarded) Eq. (1.4) can be integrated even further. We obtain

$$C_{X_0}(p_0, \vec{p}) \;\; = \;\; \int dp_0' P_{X_0}(p_0, p_0') A_\infty(p_0', \vec{p}) B_\infty(p_0', \vec{p}). \tag{1.5}$$

The convolution product of two two-point functions which are FTPF's and satisfy (1) and (2) is also FTPF. Then this product is expressed through the projection operator acting on a simple product of two FTPF given at $X_0 = \infty$. As in many applications, Eq. (1.3) has been used together with the assumption that the interaction has been switched on at $t_i \to -\infty$, an adequate comparison with it requires $X_0 \to \infty$. Then, however, $P_{X_0}(p_0, p_{0,1}) \to \delta(p_0 - p_{0,1})$. Now the integration in Eq. (1.5) becomes trivial! Thus we have just verified that the limit $X_0 \to \infty$ of the convolution product of two FTPF's satisfying (1) and (2) is equal to the lowest-order contribution in the gradient expansion (i.e., to the simple product).

We find that some quantities, obtained in low orders in the perturbative expansion, e.g., bare propagators, one-loop self-energies, belong to this class. This enables us to sum the Schwinger-Dyson series with the propagators and self-energies being FTPF. Under the conditions (1) and (2), the retarded, advanced, and Keldysh components at finite X_0 are obtained by a simple action (smearing) of the projection operator onto the corresponding quantities obtained at $X_0 = \infty$, and the convolution product is a simple multiplication:

$$\mathcal{G}_{R,X_0}(p_0, \vec{p}) = \int dp_{0,1} P_{X_0}(p_0, p_{0,1}) \mathcal{G}_{R\infty}(p_{0,1}, \vec{p}),$$
$$\mathcal{G}_{R\infty}(p_{0,1}, \vec{p}) = \frac{G_{R\infty}(p_{0,1}, \vec{p})}{1 - i\Sigma_{R\infty}(p_{0,1}, \vec{p}) G_{R\infty}(p_{0,1}, \vec{p})}, \tag{1.6}$$

and similarly for the advanced and Keldysh components.

From our study one can deduce a general rearrangement of the perturbation expansion at the non-Keldysh integration path: the contributions look like the zeroth order of the gradient expansion, with the slow coordinate (X_0) pushed to $+\infty$, but the use of PF manifests itself as the appearance of the $(\sum_j q_{0,j} + i\epsilon)^{-1}$ factor instead of $-\pi\delta(\sum_j q_{0,j})$ for each vertex, and as an overall projection (smearing) operator instead of the exact conservation of energy.

Our study suggests that the results obtained by using the Keldysh integration path ($t_i \to -\infty$) could be related to the results of our approach (t_i finite). This relation is possible at low orders of the perturbation expansion, i.e., as long as the expressions involved are the projected functions not breaking assumptions (1) and (2). Technically, the amplitudes are related by (1.1), where, for the need of this relation, the index "X_0" refers to the contributions of our approach (t_i finite) and the index "∞" refers to the corresponding lowest-order contribution in the gradient expansion in the theories with $t_i \to -\infty$.

Now, it is $\mathcal{G}_{R,\infty}(p_0, \vec{p})$ as given in Eq. (1.6) (and similar expressions for the advanced and Keldysh components and the single self-energy insertion approximation to $\mathcal{G}_{R,A,K,\infty}$) to which the previous results about the cancellation of pinching singularities[15, 16, 18, 20] and the HTL resummation[24, 25] (and also the cancellation of collinear[13, 14] and infrared singularities if the properties (1) and (2) hold at the two-loop level) apply.

As the property (1.1) is too poor to be a true "time" dependence of Green functions obtained at high enough order to realistically approximate the true evolution of the system, one is lead to examine the properties (1) and (2) and search for possible exceptions at high enough order of complexity. This exceptions could enter either directly, through the poles and cuts in the self-energy calculated at high enough order, or through the resummation process, or something else.

The time dependence of the contributions of the assumed poles in the upper energy-semiplane resembles the relaxation of out of equilibrium systems. Needless to say, to find the assumed singularities is a hard job itself.

The paper is organized as follows.

In Sec. II we give a general setup of out of equilibrium thermal field theory, with special emphasis on the distribution function and its relation to the "temperature" function.

In Sec.III we define finite time Fourier transforms, define the projection operators, and introduce the notion of projected functions,

In Sec.IV we analyze the properties of the product of two and n two-point functions. We define the analyticity assumptions (1) and (2).

In Sec. V we define a few important examples of projected functions: bare propagators and one-loop self-energies. We find that bare propagators and one-loop self-energies satisfy assumptions (1) and (2). Within this context we discuss the restriction of the inverse bare propagator on the space of projected functions. These properties are used to sum the Schwinger-Dyson series for retarded, advanced, and Keldysh components of the propagator.

Sec. VI is devoted to the modifications of Feynman rules in the coordinate and momentum space. It is indicated that, in the absence of breakdown of assumptions (1) and (2), all the energy denominators can be replaced by the delta functions.

Sec. VII is speculative. As we expect that the retarded functions at some level of the perturbation expansion do not satisfy assumptions (1) and (2), we investigate the consequences of the assumption that the retarded self-energy function possesses the pole in the upper half of the first sheet of the complex energy plane. We show that the propagator contains contributions exponentially decaying with "time" (i.e., with the slow Wigner variable).

Sec.VIII is a brief summary of the results and ideas described in the paper.

2 Setup of Out of Equilibrium

We start by assuming that the system has been prepared at some initial time $t_i = 0$ (To avoid unessential complications, we assume that the zero-temperature renormalization has already been performed). At t_i the interaction is switched on and at time t_f it is switched off (we shall take the limit $t_f \to \infty$). For $t_i < t < t_f$, the system evolves under the evolution operator [8]

$$U(t_2, t_1) = T_c[\exp i \int_c d^4x' \mathcal{L}_I(x')], \tag{2.1}$$

where c is the integration contour connecting t_1 and t_2 in the complex time plane and T_c is the contour ordering operator. We provide it with an extra property: for all times not belonging to it gives zero the contour.

The Heisenberg field operator $\phi(x)$ is obtained from the free field $\phi_{in}(x)$ in the interaction picture as

$$\phi(x) = U(t_i, t)\phi_{in}(x)U(t, t_i), \tag{2.2}$$

$$\phi(x) = T_C[\phi_{in}(x) \exp i \int_C d^4x' \mathcal{L}_I(x')], \tag{2.3}$$

where all fields at the right-hand side are in the interaction picture, and C is the contour (from $t_i + i\epsilon$ to $t_f + i\epsilon$, from $t_f + i\epsilon$ to $t_f - i\epsilon$, and, finally, from $t_f - i\epsilon$ to $t_i - i\epsilon$, with the switching-off time pushed to infinity, $t_f \to +\infty$). In the Heisenberg picture, the average values of the operators are obtained as

$$< O(t) >= Tr \rho O(t), \tag{2.4}$$

where ρ is the density operator admitting the Wick decomposition. Specially, we define the two-point Green function as

$$G^{(C)}(x, x') = -i < T_C \phi(x)\phi(x') > . \tag{2.5}$$

With the help of (2.1) it can be written as

$$G^{(C)}(x, x') = -i < T_C \left(\exp[i \int_C d^4x' \mathcal{L}_I(x')] \phi_{in}(x)\phi_{in}(x') \right) > . \tag{2.6}$$

We assume the single-particle density operator to be stationary with respect to the free Hamiltonian $H_0 = \sum_j H_j$ (for an alternative choice of initial density see Ref. [39]):

$$\rho = \sum_n |\psi_n > p_n < \psi_n| = \frac{1}{Z} \exp(-\sum_j \beta_j H_j), \tag{2.7}$$

where the "temperature" function β_j (the "temperature" of the j^{th} degree of freedom) is adopted to obtain the given initial state particle distribution. For free fermions (upper sign) or bosons (lower sign), one has

$$\sum_j \beta_j H_j = \int d^4p \beta(p_0) p_0 \Theta(p_0) \delta(p_0^2 - \vec{p}^2 - m^2) a_p^+ a_p \tag{2.8}$$

and obtains the distribution function $f(p_0)$ as a function of $\beta(p_0)$,

$$f_\beta(p_0) = \frac{1}{p_0} Tr p_0 \rho = \frac{1}{\exp \beta(p_0) p_0 \pm 1}, \tag{2.9}$$

or the inverse relation

$$\beta_f(p_0) = \frac{\log(\frac{1}{f(p_0)} \mp 1)}{p_0}. \tag{2.10}$$

The free fields are expanded in creation and annihilation operators

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} (a_p e^{-ipx} + a_p^+ e^{ipx}), \tag{2.11}$$

with $p_0 = \omega_p = (\vec{p}^2 + m^2)^{1/2}$.

$$< a_p^+ a_{p'} >= (2\pi)^3 2\omega_p f(\omega_p) \delta(\vec{p} - \vec{p}'),$$

$$< a_p a_{p'}^+ >= (2\pi)^3 2\omega_p (1 \mp f(\omega_p)) \delta(\vec{p} - \vec{p}'), \tag{2.12}$$

where $f(\omega_p)$ is the given initial distribution.

The noninteracting contour Green function is given as

$$G_{in}^{(c)}(x, x') = -i < T_c \phi_{in}(x) \phi_{in}(x') > . \tag{2.13}$$

Depending on whether the times x_0 and x_0' belong to the upper ("1") or lower ("2") part of the path C, the function $G_{in}^{(c)}(x, x')$ splits into the components $G_{\mu,\nu,in}(x, x')$, $\quad \mu, \nu = 1, 2$. For the times $x_0 < 0$ or $x_0' < 0$, the Green function is equal to zero owing to our definition of T_C.

3 Projected functions

Let us start with the two-point Green function $G(x, y)$. The quantities x and y are four-vector variables with time components in the range $t_i < x_0, y_0 < \infty$ (here t_i is the time at which we switch on the interaction; it is usually, set to $-\infty$, but we set it to $t_i = 0!$). We define the Wigner variables as usual:

$$X = \frac{x+y}{2},\ \ s = x - y,$$

$$G(x,y) = G(X + \frac{s}{2}, X - \frac{s}{2}). \tag{3.1}$$

We adopt the simplifying assumption of the homogeneity in space coordinates. This assumption excludes any dependence on $\vec{X}$.

The lower limit on x_0, y_0 implies $0 < X_0,\ \ -2X_0 < s_0 < 2X_0$. The Green function can be expressed in terms of the Fourier integral with respect to s_0, s_i:

$$G(X + \frac{s}{2}, X - \frac{s}{2}) = (2\pi)^{-4} \int d^4 p e^{-i(p_0 s_0 - \vec{p}\vec{s})} G(p_0, \vec{p}; X). \tag{3.2}$$

Here

$$G(p_0, \vec{p}; X) = \int_{-2X_0}^{2X_0} ds_0 \int d^3 s e^{i(p_0 s_0 - \vec{p}\vec{s})} G(X + \frac{s}{2}, X - \frac{s}{2}). \tag{3.3}$$

This is, in fact, the Fourier integral of the product of the function $G(X + \frac{s}{2}, X - \frac{s}{2})$ with the function $\Theta(2X_0 - s_0)\Theta(2X_0 + s_0)$, which projects any given function to its $-2X_0 < s_0 < 2X_0$ carrier. The Fourier integral can also be joined to the full function:

$$G_\infty(p_0, \vec{p}; X) = \int_{-\infty}^{\infty} ds_0 \int d^3 s e^{i(p_0 s_0 - \vec{p}\vec{s})} G(X + \frac{s}{2}, X - \frac{s}{2}). \tag{3.4}$$

Then there is a simple relation between the two transforms:

$$G(p_0, \vec{p}; X_0) = \int_{-\infty}^{\infty} dp_0' P_{X_0}(p_0, p_0') G_\infty(p_0', \vec{p}; X_0), \tag{3.5}$$

where

$$P_{X_0}(p_0, p_0') = \frac{1}{2\pi} \int_{-2X_0}^{2X_0} ds_0 e^{is_0(p_0 - p_0')} = \frac{1}{\pi} \frac{\sin\left(2X_0(p_0 - p_0')\right)}{p_0 - p_0'}, \tag{3.6}$$

and

$$e^{-is_0 p_0'} \Theta(2X_0 + s_0)\Theta(2X_0 - s_0) = \int dp_0 e^{-is_0 p_0} P_{X_0}(p_0, p_0'). \tag{3.7}$$

It is important to note that

$$\lim_{X_0 \to \infty} P_{X_0}(p_0, p_0') = \lim_{X_0 \to \infty} \frac{1}{\pi} \frac{\sin\left(2X_0(p_0 - p_0')\right)}{p_0 - p_0'} = \delta(p_0 - p_0'). \tag{3.8}$$

For those functions which are identically zero outside the interval $-2X_0 < s_0 < 2X_0$, the two transforms coincide, and relation (3.5) turns into an identity, satisfied by the Fourier integrals of the projected functions. Evidently, this leads to the hierarchy of the P_{X_0} projectors:

$$P_{X_{0m}}(p_0, p"_0) = \int dp_0' P_{X_0}(p_0, p_0') P_{X_0'}(p_0', p"_0), \quad X_{0m} = min(X_0, X_0'). \tag{3.9}$$

Note here that the ∞ in Eq. (3.5) refers to the infinite domain of integration; the function G_∞, in general, still depends on X_0.

In this paper, the projected function is a very special two-point function $F(x, y) = F(X + s/2, X - s/2)$: it is a function of $(s_0, \vec{s})$ within the interval $-2X_0 < s_0 < 2X_0$, and identical to zero outside:

$$F(X + \frac{s}{2}, X - \frac{s}{2}) = \begin{pmatrix} F(s_0, \vec{s}) & -2X_0 < s_0 < 2X_0 \\ 0 & s_0 < -2X_0 \quad and \quad 2X_0 < s_0 \end{pmatrix}. \tag{3.10}$$

The projected function still satisfies Eq.(3.5) but the important difference is that G_∞ depends on $(p_0, \vec{p})$ and not on X_0.

Important examples of projected functions are the retarded, advanced, and Keldysh components of free propagators. Further examples will emerge in the following sections.

4 Convolution Product of Two Two-Point Functions

Let us now consider the convolution product of two Green functions:

$$C = A * B \Leftrightarrow C(x, y) = \int dz A(x, z) B(z, y). \tag{4.1}$$

In terms of Fourier transforms:

$$C(p_0, \vec{p}; X) = \int_{-2X_0}^{2X_0} ds_0 \int d^3s \int d^4z e^{i(p_0 s_0 - \vec{p}\ \vec{s})}$$

$$\times \frac{1}{(2\pi)^4} \int d^4p_1 e^{-i(p_{01} s_{01} - \vec{p}_1 \vec{s}_1)} A(p_{01}, \vec{p}_1; X_1)$$

$$\times \frac{1}{(2\pi)^4} \int d^4p_2 e^{-i(p_{02} s_{02} - \vec{p}_2 \vec{s}_2)} B(p_{02}, \vec{p}_2; X_2),$$

$$X_1 = X + \frac{s_2}{2}, \ X_2 = X - \frac{s_1}{2}, \ s_1 = x - z, \ s_2 = z - y. \tag{4.2}$$

The assumed translational invariance helps us to easily integrate the space components of momenta and coordinates. To do so, we change from $d^3\vec{s} d^3\vec{z}$ to $d^3\vec{s}_1 d^3\vec{s}_2$ (Jacobian $J = 1$)

$$\vec{s} = \vec{s}_1 + \vec{s}_2, \ \vec{z} = \frac{-\vec{s}_1 + \vec{s}_2}{2} + \vec{X}. \tag{4.3}$$

The momenta should be equal ($\vec{p} = \vec{p}_1 = \vec{p}_2$) and one obtains (note that the dependence on X's is reduced to the dependence on X_0's),

$$C(p_0, \vec{p}; X_0) = \frac{1}{(2\pi)^2} \int_{-2X_0}^{2X_0} ds_0 \int dz_0 \int dp_{01} \int dp_{02}$$

$$\times e^{i(p_0 s_0 - p_{01} s_{01} - p_{02} s_{02})} A(p_{01}, \vec{p}; X_{01}) B(p_{02}, \vec{p}; X_{02}). \tag{4.4}$$

For energy integrals we proceed in a somewhat different way. We shrink our choice of functions $A(x, z)$ and $B(z, y)$ to the projected functions. Then we can use the connection to the Fourier transforms on the infinite range:

$$C(p_0, \vec{p}; X_0) = \frac{1}{(2\pi)^2} \int_{-2X_0}^{2X_0} ds_0 \int dz_0 \int dp_{01} \int dp_{02} e^{i(p_0 s_0 - p_{01} s_{01} - p_{02} s_{02})}$$

$$\times \int dp'_{01} P_{X_{01}}(p'_{01}, p_{01}) A_\infty(p_{01}, \vec{p}) \int dp'_{02} P_{X_{02}}(p'_{02}, p_{02}) B_\infty(p_{02}, \vec{p}). \tag{4.5}$$

The integration $dp'_{01}dp'_{02}$ is easily performed with the help of Eq. (3.7)and one obtains

$$C(p_0, \vec{p}; X_0) = \frac{1}{(2\pi)^2} \int_{-2X_0}^{2X_0} ds_0 \int dz_0 \int dp_{01} \int dp_{02} e^{i(p_0 s_0 - p_{01}s_{01} - p_{02}s_{02})}$$

$$\times \Theta(2X_{01} + s_{01})\Theta(2X_{01} - s_{01})\Theta(2X_{02} + s_{02})\Theta(2X_{02} - s_{02}) A_\infty(p_{01}, \vec{p}) B_\infty(p_{02}, \vec{p}). \tag{4.6}$$

The product of Θ functions is transformed into $\Theta(2X_0 + s_0)\Theta(2X_0 - s_0)\Theta(z_0)$. Then

$$C(p_0, \vec{p}; X_0) = \int\int dp_{01} dp_{02} \delta(p_0, p_{01}, p_{02}) A_\infty(p_{01}, \vec{p}) B_\infty(p_{02}, \vec{p}). \tag{4.7}$$

Here

$$\delta(p_0, p_{01}, p_{02}) = \frac{1}{(2\pi)^2} \int_{-2X_0}^{2X_0} ds_0 \int_0^\infty dz_0 e^{i(p_0 s_0 - p_{01}s_{01} - p_{02}s_{02})}$$

$$= \frac{1}{(2\pi)^2} \int_{-2X_0}^{2X_0} ds_0 \int_0^\infty dz_0 e^{i(s_0(p_0 - (p_{01}+p_{02})/2 + (z_0 - X_0)(p_{01} - p_{02})}$$

$$= P_{X_0}(p_0, (p_{01} + p_{02})/2) \frac{1}{2\pi} \frac{i}{p_{01} - p_{02} + i\epsilon} e^{-iX_0(p_{01} - p_{02})}, \tag{4.8}$$

where we have used

$$\int_0^\infty dz_0 e^{iz\alpha} = \frac{i}{\alpha + i\epsilon}. \tag{4.9}$$

We can write the final expression as

$$C(p_0, \vec{p}; X_0) = \int dp_{01} dp_{02} P_{X_0}(p_0, \frac{p_{01} + p_{02}}{2}) \frac{1}{2\pi} \frac{ie^{-iX_0(p_{01} - p_{02})}}{p_{01} - p_{02} + i\epsilon} A_\infty(p_{01}, \vec{p}) B_\infty(p_{02}, \vec{p}). \tag{4.10}$$

Expression (4.10) is the key for finite-time thermal field theory.

For further analysis,the analytic properties of FTFP as a function of complex energy are very important. We define the following properties: (1) the function of p_0 is analytic above (below) the real axis, (2) the function goes to zero as $|p_0|$ approaches infinity in the upper (lower) semiplane. The choice above (below) and upper (lower) refers to retarded (advanced) components (note here that for the Keldysh components we do not require such properties).

If A is an advanced operator satisfying assumptions (1) and (2), we can integrate expression (4.10) even further. After closing the $p_{0,1}$ integration contour in the lower semiplane, one obtains (if B is a retarded operator satisfying (1) and (2), one can achieve the same result by closing the $p_{0,2}$ integration contour in the upper semiplane):

$$C(p_0, \vec{p}; X_0) = \int dp_{01} P_{X_0}(p_0, p_{01}) A_\infty(p_{01}, \vec{p}) B_\infty(p_{01}, \vec{p}). \tag{4.11}$$

This is an extraordinary result: the convolution product of two FTPF's is FTPF under the conditions (1) and (2).

As expected, in the limit $X_0 = \infty$, Eq.(4.11) becomes a simple product

$$\lim_{X_0 \to \infty} C(p_0, \vec{p}; X_0) = A_\infty(p_0, \vec{p}) B_\infty(p_0, \vec{p}). \tag{4.12}$$

At finite X_0, Eq.(4.11) exhibits a smearing of energy (as much as it is necessary to preserve the uncertainty relations!).

4.1 Convolution Product of n Two-Point Functions

The product of n two-point functions is obtained by repeating of the above procedure:

$$C(p_0,\vec{p};X_0) = \int \prod_{j=1}^{n-1}(dp_{0j})dp_{0n}P_{X_0}(p_0,(p_{01}+p_{0n})/2)$$
$$\times \prod_{j=1}^{n-1}\left(A_{j,\infty}(p_{0j},\vec{p})\frac{1}{2\pi}\frac{i}{p_{0j}-p_{0j+1}+i\epsilon}\right)e^{-iX_0(p_{0,1}-p_{0,n})}A_{n,\infty}(p_{0,n},\vec{p}). \tag{4.13}$$

We note here: the condition that also the intermediate products should be projected functions requires that at least n-1 of the functions in the product satisfy assumptions (1) and (2) (the retarded should be on the right hand-side and the advanced on the left-hand side, and inbetween should be that function which eventually does not satisfy (1) and (2)).

Then one can integrate (index R for retarded, a similar expression for advanced)

$$I_j(p_{0,j-1},p_{0,j+1},\vec{p}) = \int dp_{0,j}\frac{1}{2\pi}\frac{i}{p_{0,j-1}-p_{0,j}+i\epsilon}A_{R,j,\infty}(p_{0,j},\vec{p})\frac{1}{2\pi}\frac{i}{p_{0,j}-p_{0,j+1}+i\epsilon}$$
$$= A_{R,j,\infty}(p_{0,j-1},\vec{p})\frac{1}{2\pi}\frac{i}{p_{0,j-1}-p_{0,j+1}+i\epsilon}. \tag{4.14}$$

Then one finds

$$C(p_0,\vec{p};X_0) = \int dp_{0,1}P_{X_0}(p_0,p_{0,1})\prod_{j=1}^{n}A_{j,\infty}(p_{0,1},\vec{p}). \tag{4.15}$$

5 Examples of Projected Functions

5.1 Propagator

We start with Eqs. (2.11), (2.12), and (2.13). The transition to the R/A basis is straightforward. Careful calculation gives for the retarded component ($0 < x_0$, $0 < x_0$):

$$G_R(x,x') = -G_{1,1}+G_{2,1} = \int d^4p\frac{-i}{p^2-m^2+2i\epsilon p_0}e^{ip(x-x')}, \tag{5.1}$$

and for the Keldysh component:

$$G_K(x,x') = G_{1,1}+G_{2,2} = \int d^4p 2\omega_p\delta(p^2-m^2)(1\mp 2f(\omega_p))e^{ip(x-x')}. \tag{5.2}$$

As our G_R and G_K depend only on $s = x - x'$ and vanish for times before switching on the interaction, they are projected functions. The Fourier transform over the infinite $x_0 - x_0'$ interval gives as usual[4, 12, 10, 20]

$$G_{R,\infty}(p) = \frac{-i}{p^2-m^2+2i\epsilon p_0},$$
$$G_{K,\infty}(p) = 2\omega_p\delta(p^2-m^2)(1\mp 2f(\omega_p)). \tag{5.3}$$

The finite Fourier transform ($x_0 > 0, x'_0 > 0, X_0 = (x_0 + x'_0)/2$) is obtained by smearing

$$G_{R,X_0}(p) = P_{X_0} G_{R,\infty}(p),\ G_{K,X_0}(p) = P_{X_0} G_{K,\infty}(p). \tag{5.4}$$

It is easy to verify that the spinor or tensor factor does not change our conclusion (5.4)! One can even integrate expression (5.4). For a scalar particle, one obtains

$$\begin{aligned} G^0_{R,X_0}(p) &= \frac{-i}{p_0^2 - \vec{p}^2 - m^2 + 2i\epsilon p_0}\left(1 - (\cos 2X_0\omega_p - i\frac{p_0}{\omega_p}\sin 2X_0\omega_p)e^{2iX_0(p_0+i\epsilon)}\right) \\ &= G^0_{R,\infty}(p_0)\left(1 - (\cos 2X_0\omega_p - i\frac{p_0}{\omega_p}\sin 2X_0\omega_p)e^{2iX_0(p_0+i\epsilon)}\right). \end{aligned} \tag{5.5}$$

In this and similar expressions the general rule is to keep the "ϵ" finite to the end of calculation. Evidently, for $X_0 \to \infty$, the first term in $G_{R,X_0}(p)$ gives $G_{R,\infty}$, while the other two "oscillate out". For the Keldysh component, one obtains (for Bose-Einstein statistics)

$$\begin{aligned} G^0_{K,X_0}(p) &= \frac{1}{\pi}(1 + 2f(\omega_p))\sum_{\lambda=-1}^{1}\frac{\sin 2X_0(p_0 - \lambda\omega_p)}{p_0 - \lambda\omega_p} \\ &= \frac{2}{\pi}\omega_p[1 + 2f(\omega_p)]\frac{-\sin 2X_0 p_0 \cos 2X_0\omega_p + \frac{p_0}{\omega_p}\sin 2X_0\omega_p \cos 2X_0 p_0}{p_0^2 - \omega_p^2}. \end{aligned} \tag{5.6}$$

For a spinor particle, one obtains

$$G^{1/2}_{R,X_0}(p) = -\gamma^0 e^{2iX_0(p_0+i\epsilon)}\frac{\sin 2X_0\omega_p}{\omega_p} + G^o_{R,X_0}(p)(\gamma^\mu p_\mu + m). \tag{5.7}$$

Similarly, for a vector particle (for simplicity, we choose the Feynman gauge):

$$G^1_{R,X_0,\mu,\nu}(p) = g_{\mu,\nu} G^0_{R,X_0}(p). \tag{5.8}$$

We note here that the explicit expressions (5.5)-(5.8) will not be necessary for further discussion.

5.2 Inverse Propagator

To define the inverse propagator, we use the results of the preceding subsection. We define the restriction of G_R^{-1} on the subspace of projected functions by

$$G^{-1}_{R,X_0}(p_0, \vec{p}) = \int dp'_0 P_{X_0}(p_0, p'_0) G^{-1}_{R,\infty}(p'_0, \vec{p}), \quad G^{-1}_{R,\infty}(p'_0, \vec{p}) = i(p^2 - m^2 + 2i\epsilon p_0). \tag{5.9}$$

This integral does not converge in the absolute sense, thus we cannot calculate the dependence of G_R^{-1} on X_0. Nevertheless, we can apply it from the left to some class of functions. For example, we can apply it formally to G_{R,X_0} to obtain $G_R^{-1} * G_R = P_{X_0}$. This equality is obtained by the simple integration over $p_{0,2}$ in the expression of the type (4.10). We cannot verify the second identity $G_R * G_R^{-1} = P_{X_0}$ directly owing to the divergence of the integrals, but we can apply it to the projected function C $G_R * G_R^{-1} * C = C$, under the only requirement that the $C_\infty(p_0, \vec{p})$ should satisfy assumptions (1) and (2) and vanishes rapidly enough at $p_0 \to \infty$ to make the integral over $G^{-1}_{R,\infty}(p_0)C_\infty(p_0)$ convergent.

5.3 One-Loop Self-Energy

To discuss the one-loop self-energy, we start with

$$\Sigma_{\mu,\nu}(x,y) = g^2 S_{\mu,\nu}(x,y) D_{\mu,\nu}(x,y)$$

$$= g^2 \int d^4p e^{-ips} P_{X_0}(p_0, p_0') S_{\mu,\nu,\infty}(p_0', \vec{p}) \int d^4q e^{-iqs} P_{X_0}(q_0, q_0') D_{\mu,\nu,\infty}(q_0', \vec{q}). \quad (5.10)$$

The Fourier transform (with respect to $s = x - y$) is

$$\Sigma_{\mu,\nu,X_0}(p_{01}, \vec{p}_1) = g^2 \int_{-2X_0}^{2X_0} ds_0 \int dp_0 d^3p dq_0 e^{-i(p_0+q_0)s_0} \Theta(2X_0 - s_0)\Theta(2X_0 + s_0)$$

$$\times S_{\mu,\nu,\infty}(p_0', \vec{p}) D_{\mu,\nu,\infty}(q_0', \vec{p}_1 - \vec{p})$$

$$= \int dp_0' d^3p dq_0' P_{X_0}(p_{01}, p_0' + q_0') S_{\mu,\nu,\infty}(p_0', \vec{p}) D_{\mu,\nu,\infty}(q_0', \vec{p}_1 - \vec{p}), \quad (5.11)$$

where as an intermediary step we have used the representation of the bare propagators (5.4) and the representation of the projectors (3.6) and (3.7). Finally, one reads (5.11) as

$$\Sigma_{R,A,X_0}(p) = P_{X_0}\Sigma_{R,A,\infty}(p), \ \Sigma_{K,X_0}(p) = P_{X_0}\Sigma_{K,\infty}(p). \quad (5.12)$$

Now, to verify whether or not the one-loop self-energy $\Sigma_{R,\infty}$ satisfies the assumptions (1) and (2), one observes that the vacuum contribution satisfies them (for exceptions, see, e.g., Ref.[41, 30]), while the contributions to $\Sigma_{R,\infty}$ from various k_0 points are linear and additive in distribution functions ([20]). Now assume the distribution function to be zero outside of the narrow vicinity of $\bar{k}_0$: $f_S(\omega_k) = \kappa\delta(\omega_k - \omega_{\bar{k}})$, $f_D(\omega_k) = 0$. Then the corresponding contribution to $\Sigma_{R,\infty}$ can be easily obtained (using (2.3), (3.3), (3.6), and (3.7) of ([20]):

$$\Delta\Sigma_{R,\infty} = \frac{g^2\kappa}{(2\pi)^3} \int d^4k \delta(\omega_k - \omega_{\bar{k}}) \frac{1}{(q_0 - k_0)^2 - (\vec{q} - \vec{k}) - m_S^2 + 2i\epsilon(q_0 - k_0)} F, \quad (5.13)$$

where the factor $F = F(k_0, |\vec{k}|, q_0, |\vec{q}|, \vec{k}\vec{q}, ...)$ includes the information about spin and internal degrees of freedom.

For finite ϵ, this contribution possesses singularities only below the real axis in the complex q_0 plane, and vanishes as $|q_0| \to \infty$ in the upper semiplane. Then, the same conclusion is also valid for the sum of such a type of contributions which, in the limit, constitute the arbitrary $f_S(\omega_k)$ distribution function. With little effort one proves the same for the arbitrary $f_D(\omega_{k-q})$. Thus one may conclude: the retarded one-loop self-energy is the projected function and satisfies the assumptions (1) and (2). But there is no guarantee that the imaginary part of $\Sigma_{R,\infty}$ is positive.

5.4 Resummed Schwinger-Dyson Series

To sum the Schwinger-Dyson series, we assume that the functions $A_\infty(p_0, \vec{p})$ appearing in (4.13) as functions of p_0 satisfy the requirements (1) and (2) for the retarded functions in the upper and for the advanced in the lower semiplane.

In such a case, if the retarded function is real between the cuts on the part of the real axis, the Schwartz theorem tells us that the same is valid in the lower semiplane of the first Riemann sheet.

For the retarded bare propagators, our assumption is valid.

At equilibrium, perturbation theory yields the full propagator as a set of Fourier coefficients. The analytic continuation in the energy plane is not unique. This freedom is used to choose an analytic continuation that satisfies the requirements (1) and (2) defined in Sec.IV. The positivity property of the spectral density then implies that the propagator has neither zeroes nor poles off the real axis[8]. Further implication is that the exact self-energy $\Sigma_R(p_0,\vec{p})$ at equilibrium satisfies the properties (1) and (2), too. This is not guaranteed for approximate expressions for self-energy.

Now it is easy to write down the resummed Schwinger-Dyson series for the retarded propagator with exact full self-energy (or any other self-energy obtained by the perturbation expansion that satisfies our assumptions).

In the expression for the retarded one-loop self-energy resummed propagator, the factors G_R and Σ_R alternate regularly. This fact implies that not $\Sigma_{R,\infty}$, but $G_{R,\infty}\Sigma_{R,\infty}$ is subject of assumptions (1) and (2).

In terms of the corresponding propagator calculated at $X_0=\infty$:

$$\mathcal{G}_R(p_0,\vec{p};X_0)=\int dp_{0,1}P_{X_0}(p_0,p_{0,1})\mathcal{G}_{R,\infty}(p_{0,1},\vec{p}), \tag{5.14}$$

where $\mathcal{G}_{R,\infty}(p_{0,1},\vec{p})$ is given by Eq. (1.6). To sum the advanced component, we use the properties of the lower semiplane, in accord with our assumption.

$$\mathcal{G}_A(p_0,\vec{p};X_0)=\int dp_{0,1}P_{X_0}(p_0,p_{0,1})\mathcal{G}_{A,\infty}(p_{0,1},\vec{p}). \tag{5.15}$$

Some more job is necessary to calculate the Keldysh component. First, one inserts the analytic expression for $P_{X_0}(p_0,(p_{0,1}+p_{0,2})/2)$. Second, one integrates all retarded (advanced) components by closing the contour in the upper (lower) semiplane. One integration remains, the remaining factors recombine into $P_{X_0}(p_0,p_{0,1})$, and one obtains

$$\mathcal{G}_K(p_0,\vec{p};X_0)=\int dp_{0,1}P_{X_0}(p_0,p_{0,1})\mathcal{G}_{K,\infty}(p_{0,1},\vec{p}). \tag{5.16}$$

where[20]

$$\mathcal{G}_{K,\infty}(p_0,\vec{p})=\mathcal{G}_{A,\infty}(p_0,\vec{p})\left(h(p_0)(G^{-1}_{A,\infty}(p_0,\vec{p})-G^{-1}_{R,\infty}(p_0,\vec{p}))+i\Omega_\infty(p_0,\vec{p})\right)\mathcal{G}_{R,\infty}(p_0,\vec{p})), \tag{5.17}$$

and $h(p_0)=-\epsilon(p_0)(1\pm f(p_0))$. Note here that, in order to obtain (5.16), no assumption is made on the analytic properties of the Keldysh component of the one-loop self-energy $\Omega=\Sigma_{1,1}+\Sigma_{2,2}$. However, in addition to individual terms, the sum $\mathcal{G}_{R,\infty}(p_0,\vec{p})$ should also satisfy (1) and (2) (i.e., the imaginary part of the $\Sigma_{R,\infty}$ should be positive, what is not granted)! At this point one should cautiously consider the use of the "physical" gauge[42], in order to prevent eventual gauge artifacts.

Some indication that, in some cases, $\mathcal{G}_{R,\infty}(p_0,\vec{p})$ does satisfy assumptions (1) and (2) comes from the HTL limit. Indeed, at equilibrium, the HTL limit of $\mathcal{G}_{R,\infty}(p_{0,1},\vec{p})$ must satisfy (1) and

(2), as it is easy to verify. As the properties of density functions enter only through the thermal mass and the position of isolated poles, the same must be true for any distribution allowing the HTL approximation!

The final results (5.14) ,(5.15), and (5.16) is what one expects for the initial state in thermal equilibrium.

Out of equilibrium one expects more that just the smearing of the Green function. Clearly, as long as $\mathcal{G}_{R,\infty}(p_0,\vec{p})$ satisfies (1) and (2), all information about physics comes just from one line in the (s_0,X_0) plane. The only possible source of different behavior is the self-energies and the resummed Schwinger-Dyson series not satisfying the assumptions (1) and (2). The possibility that the out of equilibrium $\Sigma_{R,\infty}(p_0,\vec{p})$ POSSESSES POLES in the upper semiplane is intriguing. We discuss this case in Sec. VII!

6 Modifications of the Feynman rules

The calculations performed so far already contain all of the modifications of the Feynman rules required by the finite t_i assumption.

In the coordinate space, the only modification is: the bare propagators [Eqs. (5.1) and (5.2)] are limited by $0 < x_0$ and $0 < x_0'$; thus they are projected functions.

In the energy-momentum space, the above change reflects in the change of propagators, vertices, and the overall factor.

To transform to energy-momentum space, we choose some vertex j, arrange the orientation so that all lines i become outgoing, and use the propagators represented by Eqs. (5.1), (5.2), and (5.3) (the p_i momentum is joined to the line i). Exponentials attached to x_j are easily integrated with the help of Eq.(4.9):

$$\frac{1}{2\pi}\int_0^\infty dx_j e^{-ix_j\sum_i p_i} = \frac{i}{2\pi(-\sum_i p_i + i\epsilon)}. \tag{6.1}$$

After performing this integration, instead of the bare propagators we obtain their $X_0 \to \infty$ limits [Eq. (5.3)], which are the familiar propagators of the usual $(t_i \to -\infty)$ theory.

In the vertices the usual energy conserving $\delta(\sum_i p_{0,i})$ is substituted by $(2\pi)^{-1}(-\sum_i p_{0,i} + i\epsilon)^{-1}$.

Under the momentum integrals there is a leftover factor at the vertices j_A with amputated legs:

$$e^{-i\sum_{j_A} x_{j_A}(\sum_{i_{j_A}} \lambda_{i_{j_A}} p_{i_{j_A}})}, \tag{6.2}$$

where $\lambda = \pm$ depends on whether the momentum is outgoing or incoming to the vertex j_A, and i_{j_A} is running through the nonamputated lines.

The overall factor in the case of two-point functions is treated in a simple way: introduce a slow Wigner variable as an average over the times of boundary vertices, and the relative time [Eq. (3.1)]. Finally, one can Fourier transform over the relative time. There emerges an overall energy-smearing factor $P_{X_0}(p_0,p_0')$ for two-point functions and similarly for n-point functions. In the case of n-point functions, the choice of variables is large and might not be unique; namely, depending on the diagram calculated, one chooses the most appropriate set of variables.

The overall factor takes care of uncertainty relations: the larger the elapsed "time" X_0, the smaller the energy smearing.

The vertex factor contains more information: in this factor the energy is not explicitly conserved. To see what it means, assume, for a moment, that at least one of the unspecified propagators D_∞ related to the chosen vertex is retarded, $D_{R,\infty}(p_{0,j}, \vec{p}_j)$. In this case, one can integrate over q_0, close the integration path from above (owing to $e^{iX_0 p_{o,j}}$, closing from below is out of question), and collect the contributions from singularities. If there are no singularities (and we know that conditions (1) and (2) are valid for the bare propagators), one obtains just the energy conservation condition $\delta(\sum_i p_{0,i})$. The same is achieved with the outgoing momenta and advanced components of the propagator with closing the integration path from below.

Thus, to integrate the energy denominators $\sum_i p_{0,i}$, we have to inspect explicit expressions in an arbitrary diagram.

Each individual denominator $(\sum_i p_{0,i} - i\epsilon)^{-1}$ (the lines are all oriented out) can be easily integrated. To demonstrate this, we have to sum over the indices of the corresponding vertex. We rename the basis $(i,j), i,j = 1,2$ into (μ,ν), where $\mu,\nu = -1$ correspond to $i,j = 2$, and $\mu,\nu = 1$ correspond to $i,j = 1$. Then we find (we assume a three-point vertex, but the proof extends easily to any number of points) the relations of the type:

$$D_{\mu,\nu} = \frac{1}{2}(D_K - \mu D_R - \nu D_A). \tag{6.3}$$

The sum over the indices in the chosen vertex (S - , D -, T - propagators of the outgoing lines; the factor μ for the negative coupling of type-2 vertices) is

$$\sum_\mu \mu S_{\mu,\lambda} D_{\mu,\rho} T_{\mu,\nu} = \frac{1}{4}\Big(S_R D_R T_R + (S_K + \lambda S_A)(D_K + \rho D_A) T_R$$
$$+ (S_K + \lambda S_A) D_R (T_K + \rho T_A) + S_R (D_K + \rho D_A)(T_K + \rho T_A)\Big). \tag{6.4}$$

Expression (6.4) contains only terms including at least one retarded propagator: S_R, or D_R, or T_R.

Thus one can integrate the terms separately to obtain that the factor $(\sum_i p_{0,i} - i\epsilon)^{-1}$ is effectively replaced by $i\pi\delta(\sum_i p_{0,i})$.

As there is nothing special at this vertex (the indices λ, ρ, ν remain unspecified) one may conclude that this is a general feature. Nevertheless, one should do it very cautiously, step by step, while the desired "heretic" singularities may appear at some degree of complexity. The diagrams with resummed self-energy subdiagrams are particularly sensitive. In this case, one is strongly advised to undertake the intermediate step: the Fourier transform of the two-point function with respect to the relative time, investigate the analytic structure, and then perform the multiplication of two-point functions. Then, as seen in Eqs. (4.10) and (4.13), we find a new element in addition to the energy denominator $(-p_{0,j} + p_{0,j+1} - i\epsilon)^{-1}$. One obtains the extra factor $e^{-iX_0(p_{0,j} - p_{0,j+1})}$. This factor turns the singularities of the retarded function in the upper semiplane from "heretic" to desired (however, this does not apply to the singularities present already at equilibrium, these are pathologies of the approximations done and should be eliminated[41, 30]), as we shall see in the next section.

7 Poles in the Upper Semiplane

The search for singularities of the retarded functions situated in the upper semiplane is, certainly, outside the scope of this paper. Nevertheless, it is important to find whether they destroy the theory, as it would happen if they existed at equilibrium, or whether they contribute to its relevance, as we expect within this formalism. Here we adopt the simplest possibility that the singularities of $\mathcal{G}_R(p_0,\vec{p})$ and $\Sigma_{R,\infty}(p_0,\vec{p})$ are just simple poles at the points $\bar{p}_0$ in the upper semiplane. The pole contribution to the Green function

$$\mathcal{G}_\infty(p_0) = \frac{a}{p_0-\bar{p}_0}, \tag{7.1}$$

can be projected to finite X_0:

$$\mathcal{G}_{X_0}(p_0) = a\epsilon(Im\bar{p}_0)\frac{1-e^{-2iX_0(p_0-\bar{p}_0)\epsilon(Im\bar{p}_0)}}{p_0-\bar{p}_0}, \tag{7.2}$$

and Fourier transformed back to variables (X_0, s_0):

$$\mathcal{G}(X_0+\frac{s_0}{2}, X_0-\frac{s_0}{2}) = iae^{-i\bar{p}_0 s_0}\left(\Theta(s_0\epsilon(Im\bar{p}_0)) - \Theta(-s_0\epsilon(Im\bar{p}_0)-2X_0)\right). \tag{7.3}$$

Evidently this contribution is a projected function. For $\epsilon(Im\bar{p}_0)=-1$, it satisfies assumptions (1) and (2) as a retarded function, but not as an advanced function (and for $\epsilon(Im\bar{p}_0)=1$, just the opposite).

Eq. (7.2) exhibits the exponential decay $e^{-2X_0|Im\bar{p}_0|}$ for an arbitrary sign of $Im\bar{p}_0$.

Now we assume that it is $\Sigma_{R,\infty}(p_0,\vec{p})$ where a pole at $\bar{p}_0$appears with $\epsilon(Im\bar{p}_0)=1$. In this case, the relation (4.14) changes into

$$I_j(p_{0,j-1},p_{0,j+1},\vec{p}) = \Sigma_{R,\infty}(p_{0,j-1},\vec{p})\frac{1}{2\pi}\frac{i}{p_{0,j-1}-p_{0,j+1}+i\epsilon}$$
$$-2\pi i \lim_{p_{0,j}\to\bar{p}_{0,j}}\left(p_{0,j}\Sigma_{R\infty}(p_{0,j},\vec{p})\right)\frac{1}{2\pi}\frac{i}{p_{0,j-1}-\bar{p}_{0,j}+i\epsilon}\frac{1}{2\pi}\frac{i}{\bar{p}_{0,j}-p_{0,j+1}+i\epsilon}. \tag{7.4}$$

In the expression for the resummed Schwinger-Dyson retarded propagator, the product of n ($n=2l+1$ is odd!) two-point functions contains bare propagators and one-loop self-energies, arranged in alternating order. As the methods developed in Sec. III allow the calculation of products in which only one function possesses "heretic" poles, we can integrate only the $n=3$ contribution (but not $n=5,7,...$):

$$\mathcal{G}_R^{(3)} = G_R * \Sigma_R * G_R,$$

$$\mathcal{G}_R^{(3)}(p_0,\vec{p},X_0) = \int dp_{0,1}P_{X_0}(p_0,p_{0,1})G_{R\infty}(p_{0,1},\vec{p})\Sigma_{R\infty}(p_{0,1},\vec{p})G_{R\infty}(p_{0,1},\vec{p})$$

$$+\frac{\pi}{\omega_p}\sum_{\lambda=-1}^{+1}\lambda F(\lambda\omega_p)$$

$$\times\lim_{\bar{p}_0\to\bar{p}_0}\left(\bar{p}_0-\bar{p}_0)\Sigma_{R\infty}(\bar{p}_0,\vec{p})\right)G_{R\infty}(\bar{p}_0,\vec{p})P_{X_0}(p_0,\frac{\lambda\omega_p+\bar{p}_0}{2})\frac{e^{-2iX_0(\lambda\omega_p-\bar{p}_0)}}{\lambda\omega_p-\bar{p}_0+i\epsilon}, \tag{7.5}$$

where by F we have indicated the spinor or tensor factor in the bare propagator[20]. Now it is easy to recognize the first term in Eq.(7.5) as the second term in the expansion of Eq.(5.14). Evidently, the first term in Eq.(7.5) is FTPF, but does not satisfy assumptions (1) and (2). The second term in Eq.(7.5) is the contribution from the pole of $\Sigma_{R\infty}$ situated in the upper semiplane. It is not FTPF. The dependence on X_0 is significant: the exponential "time" decay $e^{-2X_0 Im\bar{p}_0}$ is the expected type of corrections to the contributions from $X_0 \to \infty$.

Note here that the corresponding contribution at equilibrium would not produce any exponentially decaying contribution. Indeed, the sum of the Schwinger-Dyson series would lead to the expression $(1 - \Sigma_{R,\infty} G_{R,\infty})^{-1} - 1$, which is regular at $\bar{p}_o$ and the residue vanishes!

This feature is not to be expected out of equilibrium; more probably, owing to the appearance of the non-FTFP type of functions, the resummed Schwinger-Dyson retarded propagator will contain more terms of the exponential decay type.

When $\mathcal{G}_{R,\infty}(p_0, \vec{p})$ possesses the pole at $\bar{p}_0$, such that $\epsilon(Im\bar{p}_0) = 1$, and $\mathcal{G}_{K,\infty}(p_0, \vec{p})$ satisfies assumptions (1) and (2), we write $\mathcal{G}_{R,\infty}$ (and $\mathcal{G}_{A,\infty}$) as a sum of the term satisfying assumptions (1) and (2) and the pole term. Then the Keldysh component $\mathcal{G}_K = \mathcal{G}_A * G_K * \mathcal{G}_R$, will consist of four terms. One term satisfying assumptions (1) and (2), two terms with single-pole contributions, and one term with a double-pole contribution (i.e., from the pole terms in $\mathcal{G}_{A,\infty}$ and $\mathcal{G}_{R,\infty}$). The first three terms exhibit the behavior already seen in Eq. (7.5). Because of the presence of at least two functions (i.e., $\mathcal{G}_A$ and $\mathcal{G}_R$) not satisfying (1) and (2), the last term cannot be evaluated using the methods exposed above.

8 Summary

We consider out of equilibrium thermal field theories with switching on the interaction occurring at finite time ($t_i = 0$). We study Fourier transforms (also in the relative time s_0) of two-point functions.

To develop a calculation scheme based on first principles, we define a very useful concept of projected functions: a two-point function with the property that it is zero for $x_o < t_i$ and for $y_0 < t_i$; for $t_i < x_0$ and $t_i < y_0$, the function depends only on $x_0 - y_0$. We find that many important functions are of this type: bare propagators, one-loop self-energies, resummed Schwinger-Dyson series with one-loop self-energies, etc.

The properties of the Fourier transforms are particularly simple if they satisfy the analyticity assumption: (1) The function of p_0 is analytic above (for a retarded function, below for an advanced function) the real axis. (2) The function goes to zero as $|p_0|$ approaches infinity in the upper (lower) semiplane. We find that these assumptions are very natural at low orders of the perturbation expansion.

The convolution product of projected functions is remarkably simple, much simpler than what one would expect from the gradient expansion.

The Schwinger-Dyson series, with bare propagators and self-energies being projected functions satisfying assumptions (1) and (2), is resummed in closed form without the need for the gradient expansion.

The Feynman diagram technique is reformulated: there is no explicit energy conservation at vertices, there is an overall energy-smearing factor taking care of the finite elapsed time (X_0) and the uncertainty relations.

The relation between the amplitudes (valid at low orders of the perturbation expansion) of the theory with switching on the interaction in the remote past and the theory with finite switching-on time, enables one to rederive the results such as cancellation of pinching singularities, cancellation of collinear and infrared singularities, HTL resummation, etc. Previously, these results were considered applicable only to the lowest-order contributions in the gradient expansion.

Relaxation phenomena enter through the assumed singularities in the upper semiplane (for retarded functions, in the lower semiplane for advanced functions). In equilibrium theory, such singularities would cause a disaster i.e., the contributions growing exponentially with time. In our approach, thanks to the extra factor in the vertex function, these singularities contribute terms dying out with "time" (slow Wigner variable) as $e^{-2X_0 Im\bar{p}_0}$, as expected for relaxation processes.

It is very important to find the nature and positions of the assumed singularities but this is certainly outside the scope of this paper.

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